STABILITY OF LAMINAR LIQUID FLOW IN RECTANGULAR PIPES

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The critical Reynolds numbers and maximal instability locations of laminar liquid flow in rectangular pipes are obtained theoretically as a function of the rectangle side ratio. The results are confirmed experimentally.

The expression for the critical Reynolds number R for flow in a straight pipe [1] in dimensionless form is

$$r \equiv \frac{R_*}{R_*^0} = \frac{2}{3^{3/2}} \frac{\langle u \rangle}{\sup_F \{u \,|\nabla_{\xi\eta} u|\}} \quad \text{for } \Delta_{\xi\eta} u = -1, \qquad u|_\Gamma = 0 \tag{1}$$

and the local instability is minimal where $\sup_F\{u|\nabla_{\xi\eta} u|\}$ is reached. Here $x = s\xi$, $y = s\eta$ are Cartesian coordinates at section F of the pipe (with controur Γ); s is the hydraulic radius, angle brackets denote averaging over F, the asterisk corresponds to loss of stability; the index 0 corresponds to a circular pipe (in the following we assume $R_*^0 = 2300$ [2, 3], which corresponds to an overall pipe criterion q_* [1] of about 885).

For rectangular pipes ($|x| \leq a$, $|y| \leq b$) the shape F is characterized by the parameter $k = b/a$ so that $r = r(k)$. Without limiting generality we shall assume $k \leq 1$ and examine the first quadrant $0 \leq x \leq a$, $0 \leq y \leq b$. In this case the boundary conditions take the form

$$u\left(\frac{1+k}{2k},\ \eta\right) = u\left(\xi,\ \frac{1+k}{2}\right) = \frac{\partial u}{\partial \xi}(0,\ \eta) = \frac{\partial u}{\partial \eta}(\xi,\ 0) = 0 \tag{2}$$

The quantity $u|\nabla_{\xi\eta} u|$, in addition to its absolute maximum $\sup_F\{u|\nabla_{\xi\eta} u|\}$ for $x=0$, $y=y_*$ $(0 < y_* < b$; we assume $k < 1)$, may also have a relative maximum for $y=0$, $x=x_*$ $(0 < x_{**} < a)$. Correspondingly, the absolute and relative minima of $\langle u\rangle\{u|\nabla_{\xi\eta} u|\}^{-1}$ lead to the Reynolds numbers R_* (critical) and $R_{**} > R_*$, corresponding to the onset of instability at the noted points; R_*, R_{**}, y_*, and x_{**} as functions of k are to be determined.

The appearance of the second of the instability zones $x \sim 0$, $y \sim y_*$ $(R \geqslant R_*)$; $x \sim x_{**}$, $y \sim 0$ $(R \geqslant R_{**})$, serving as an additional isolated turbulization source, in the case of an extended laminar regime causes marked increase (the larger, the larger is k) of the probability of general turbulization of the flow. The situation for an annular tube was similar [1].

In contrast with the pipe sections examined previously [1] in the rectangular pipe case there is no finite analytic expression for u [3]. The calculation becomes correspondingly more complicated, primarily owing to the need for numerical integration of (1).

The numerical solution of the Poisson equation (1) by the difference method using the boundary conditions (2) with subsequent calculation of $\langle u\rangle\{u|\nabla_{\xi\eta} u|\}^{-1}$ at the grid nodes and selection of the minimal value was carried out on a Minsk-22 computer. To obtain the grid each half of the rectangle sides was divided for $k > 0.1$ into 32 equal parts; for $k \leq 0.1$ they were divided respectively into 85 and 24 parts. The iterations were terminated as soon as the discrepancy in u became less than $2^{-23} \approx 1.2 \cdot 10^{-7}$; in this case $0.08 \leqslant \langle u\rangle \leqslant 0.14$, $0.12 \leqslant u_{max} \leqslant 0.29$. The computation of each version with a definite initial value of k required about three hours. Twenty versions were computed in all. The basic results of the calculation are shown in Figs. 1, 2, and 3.

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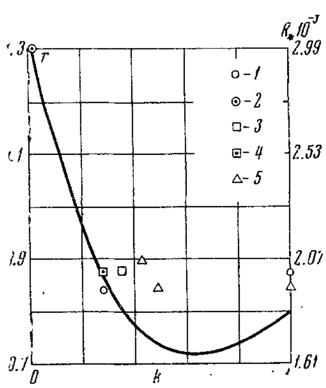

Fig. 1

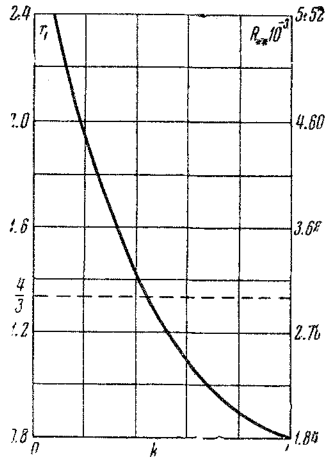

Fig. 2

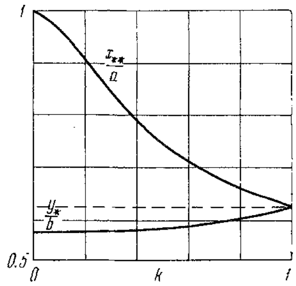

Fig. 3

The quantity $r(k)$ decreases from the maximal value $r(0) = {}^4/_3$ (flat channel) [1] to the minimal value $r(0.6255) \approx 0.7186$ and then increases somewhat to $r(1) \approx 0.7981$ (square section). It appears that the flow in the pipe is more stable (r is larger), the more uniform is the distribution of u in F (i.e., the more the measurements of the rectuangular section differ) or the more nearly symmetrical is the distribution (the closer the rectangle is to a square). Therefore we can expect, for example, that for sections in the form of regular n-gons ($n \geq 3$) $r(r < 1$ [4]) is a monotonically increasing function of n which approaches 1 as $n \to \infty$.

The value $r = r(1)$ is also reached for $k \approx 0.366$. Thus the inverse dependence $r(k)$ on the segment $0.366 \leq k \leq 1$ is two-valued. We note that $r = 1$ ($R_* = R_*^0$) for $k \approx 0.182$.

Figure 1 also shows the experimental points of Schiller [5], Davies and White [6], Cornish [7], Nikuradse [4], and Lea and Tadros [8], denoted by numerals 1-5. The points are taken directly from the curves of pipe resistance versus R obtained by the experimenters with account for the influence of the initial disturbance intensity and the entrance segment length [3]. However, for more intense and diverse initial disturbances R_* would be somewhat smaller. Considering this, we can state that the agreement between the experimental and theoretical results is fairly good.

The function $r_1(k) = R_{**}/R_*^0$ decreases monotonically to the value $r_1(1) = r(1)$. We note that $r_1 = 1$ and 4/3 respectively for $k \approx 0.660$ and 0.440.

The relative distance (y_*/b) between the absolute minimum stability point and the center of the rectangle is always less than the relative distance (x_{**}/a); y_*/b with increase of k increases monotonically from $1/\sqrt{3} \approx 0.5773$ for the flat channel [1] with $k = 0$ to 0.625 for the square tube with $k = 1$. Correspondingly, x_{**}/a decreases monotonically from 1 to 0.625.

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